Question 1:

① $e^{j\frac{3\pi}{4}n} = e^{j2\pi(\frac{3}{8})n}$

$\boxed{c}$ period = 8

② $x_a(t) \rightarrow \Omega_s$

$x_a(2t) \rightarrow$ double bandwidth

$\boxed{a}$ $2\Omega_s$

③ $\boxed{d}$

④ $\omega = \frac{\Omega}{F_s}$

$\Rightarrow \cos(1200\pi t) \Rightarrow \omega = \frac{1200\pi}{8000} = 0.15\pi$

$\rightarrow \cos(0.15n\pi)$

$\cos(17200\pi t) \Rightarrow \omega = \frac{17200\pi}{8000} = 2.15\pi$

$\rightarrow \cos(2\pi n + 0.15\pi n) = \cos(0.15\pi n)$

$\boxed{c}$

⑤ $\boxed{a}$ Low-pass filter (Moving Averager)

⑥ $\boxed{d}$ FIR Low-pass

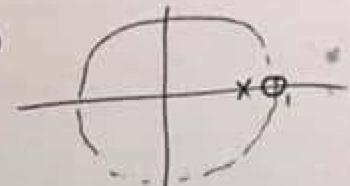
⑦ $\boxed{d}$ A linear-phase response

⑧ $\boxed{c}$ Convolving

⑨ $\boxed{b}$ 90°

⑩ $\omega = \frac{\Omega}{F_s} \Rightarrow F_s = \frac{20\pi}{\pi/5} = 100\ Hz$ $\boxed{c}$

⑪



$\boxed{c}$ high-pass filter

⑫ $\omega = \frac{5K}{10K}\pi = \frac{1}{2}\pi$

since $r_1 = r_2$

$|H| = 1$ $\boxed{a}$

⑬ $\boxed{c}$ autocorrelation of $x(n)$

⑭ $\boxed{a}$

⑮ All-pass: B, A

Minimum-phase: C

stable: A, B, C

IIR: B, C

FIR: A

⑯ pole $s = -10 \rightarrow z = e^{-10T} = e^{-2}$ $\Rightarrow H(z) = \frac{10(0.2)}{1 - e^{-2}z^{-1}}$

$\boxed{a}$ $h(n) = 2e^{-2n}u(n)$

⑰ $s = \frac{2}{0.2}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$ substitute in $H(s)$

$\boxed{a}$ $\frac{1}{2}(1 + z^{-1})$

⑱

$$H(z) = 3 + \frac{5z^{-1}}{1+0.5z^{-1}} + \frac{2(1+z^{-1})}{1-0.8z^{-1}} \quad , \quad H(z) \text{ is causal} \quad |z| > 0.8$$

$$h(n) = 3\delta(n) + 5\left(\frac{-1}{2}\right)^{n-1} u(n-1) + 2(0.8)^n u(n) + 2(0.8)^{n-1} u(n-1)$$

Question 4: [20 marks]

(a) Compare between the impulse invariance and bilinear transformation methods in terms of their advantages and their disadvantages. [4pts]

Impulse Invariance $\xrightarrow{\text{disadv.}}$ Introduce aliasing

linear mapping

Bilinear Transformation - Avoid aliasing (adv.)

- non linear distortion (disadv.)

(b) Given the following specifications of a digital filter:

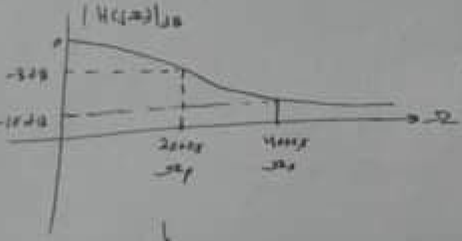
- Lowpass filter: 0 – 10 KHz passband
- Maximum passband ripple: -3dB (at 10 KHz)
- Sampling frequency: Fs = 100 KHz
- Transition band: 10 KHz to 20 KHz
- Stopband attenuation: 10dB (starting at 20 KHz)
- The filter must be monotonic in the pass and stop bands, (i.e. no ripple).

An IIR filter, meets the above specifications, is required to be designed using Butterworth model and mapped to digital using bilinear transform.

i) Draw an estimate for magnitude frequency response of the specified digital filter and its equivalent analogue filter. Clearly specify axis labels and all necessary values on the graphs. [4pts]

$\Omega_p = 2\pi f_p = 20000\pi$

$\Omega_s = 2\pi f_s = 40000\pi$



$\omega_p = 2\tan^{-1}\frac{\Omega_p T}{2}$

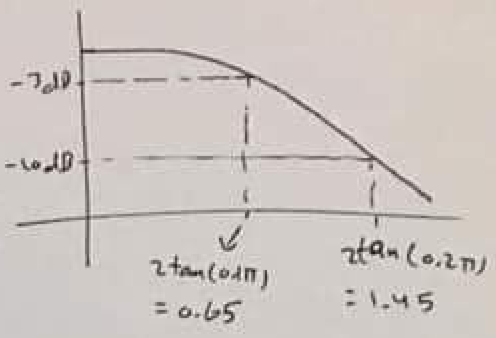
$= 2\tan^{-1}\frac{20000\pi/100000}{2} = 0.6088 \approx 0.1936\pi$

$\omega_s = 2\tan^{-1}\frac{\Omega_s T}{2}$

$= 2\tan^{-1}\left(\frac{40000\pi/100000}{2}\right) = 1.122 \approx 0.357\pi$

$F_s = 100k$, $\Omega_p = \Omega_c = 10k$, $\Omega_s = 20k$

$\omega_c = \omega_p = \frac{10k}{100k} 2\pi = 0.2\pi$, $\omega_s = 0.4\pi$



$$20\log_{10}|H(j(1.45))| \le -10$$

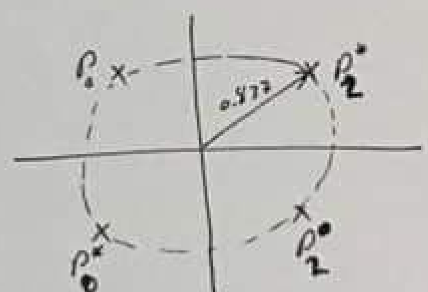
$$\Rightarrow 1 + \left(\frac{1.45}{\omega_c}\right)^{2N} = \left(10^{\frac{10}{20}}\right)^2$$

$$\Rightarrow \left(\frac{1.45}{0.65}\right)^{2N} = 9 \Rightarrow (4.976)^{2N} = 9 \Rightarrow N = \log_{4.976} 9 = 1.369 \Rightarrow \boxed{N=2}$$

$$\Rightarrow 1 + \left(\frac{1.45}{\omega_c}\right)^4 = 10 \Rightarrow \frac{\omega_c^4}{4.42} = \frac{1}{9} \Rightarrow \boxed{\omega_c = 0.837}$$

$N = 2 \Rightarrow$ 4 poles

$$P_K = \omega_c e^{j\left(\frac{\pi + 2\pi K}{N}\right)} e^{j\frac{\pi}{2}}$$



$$P_0 = 0.837\, e^{j\frac{3\pi}{4}}$$

$$P_0^* = P_1^* = 0.837\, e^{-j\frac{3\pi}{4}}$$

$$P_2^* = 0.837\, e^{-j\frac{\pi}{4}}$$

$$P_2^* = P_3^* = 0.537\, e^{j\frac{\pi}{4}}$$

$$H_a(s) = \frac{(\omega_c)^N}{\prod_1^N (s - P_k)} = \frac{(0.537)^2}{(s - 0.837e^{j\frac{3\pi}{4}})(s - 0.537\, e^{-j\frac{3\pi}{4}})}$$

$$= \frac{0.7}{(s^2 - 2(0.837)\cos(\frac{3\pi}{4})s + (0.837)^2)} = \frac{0.7}{s^2 + 1.184s + 0.7}$$

$$H(z) = \frac{0.7}{\frac{4(1-2z^{-1}+z^{-2})}{(1+z^{-1})^2} + \frac{2.368(1-z^{-1})}{(1+z^{-1})} + 0.7} = \frac{0.7(1+z^{-1})^2}{2.332z^{-2} - 9.4z^{-1} + 7.068}$$

(i) Find minimum order of the designed IIR filter, N, and the corresponding cut-off frequency, Ω_c? Show all steps. [6pts]

$$10\log_{10}\left(|H(\Omega_p)|^2\right) = \frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2N}} \le -3 \Rightarrow 1+\left(\frac{2\pi \times 0.1\pi}{\Omega_c}\right)^{2N} = 10^{0.3} \quad \cdots (1)$$

$$10\log_{10}\left(|H(\Omega_s)|^2\right) = \frac{1}{1+\left(\frac{\Omega_s}{\Omega_c}\right)^{2N}} \ge -10 \Rightarrow 1+\left(\frac{4 \times 0.4\pi}{\Omega_c}\right)^{2N} = 10^{1} \quad \cdots (2)$$

Divide (1) by (2) and take log:

$$N = \frac{\log\left[(10^{0.3}-1)/(10^{1}-1)\right]}{2\log\left(\frac{\Omega_p}{\Omega_s}\right)} \approx \boxed{2}$$

Substitute $N=2$ in eq. (1) to get $\Omega_c = \boxed{0.7365}$

(ii) Find system function Ha(s) of the designed analogue filter and the corresponding digital filter H(z)? [6pts]

We find poles of $H(s)H(-s)$ (4 poles)

$$S_k = \Omega_c\, e^{j\left(\frac{\pi+2\pi k}{2N}\right)} \cdot e^{j\frac{\pi}{2}}, \quad k = 0, 1, 2, 3$$

$$S_0 = \Omega_c e^{j\frac{3\pi}{4}} = \Omega_c e^{j135^\circ} = -0.5208 + j0.5208$$

$$S_1 = \Omega_c e^{j\frac{5\pi}{4}} = \Omega_c e^{j225^\circ} = -0.5208 - j0.5208$$

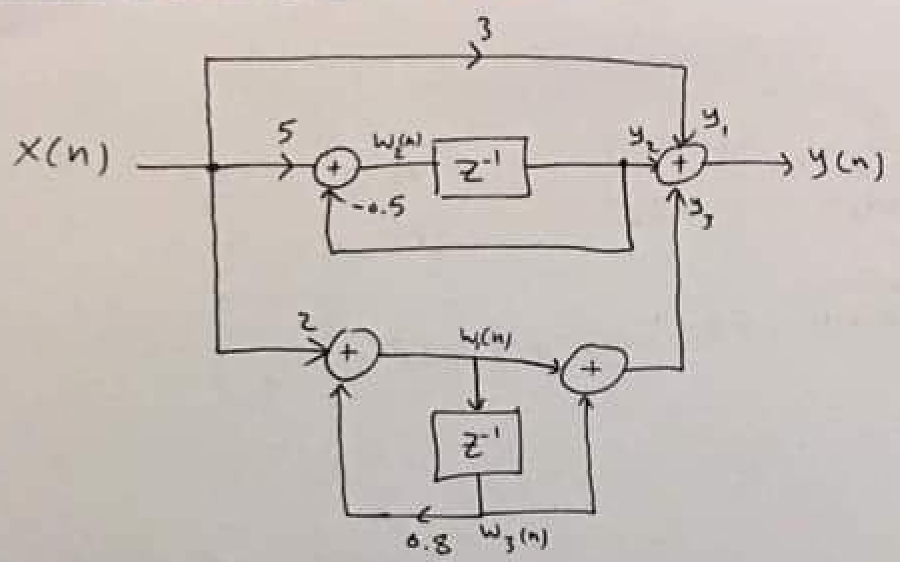
$$S_2 = \Omega_c e^{j315^\circ}$$

$$S_3 = S_2^*$$

Since S_0 and S_1 are in the left of $j\Omega$ axis $\Rightarrow S_0, S_1$ corresponds to $H(s)$ and $S_2, S_3 \to H(-s)$

$$H(s) = \frac{\Omega_c^2}{(s-s_0)(s-s_1)} \Rightarrow s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \Rightarrow \text{Find } H(z)$$

$$H(z) = \frac{0.553 + 1.102\, z^{-1} + 0.552\, z^{-2}}{1 - 0.894\, z^{-1} - 0.315\, z^{-2}}$$



$y_1 = 3x(n) \Rightarrow Y_1(z) = 3X(z)$

$w_2(n) = 5x(n) + -0.5y_2$, $y_2 = w_2(n-1) \Rightarrow Y_2(z) = z^{-1}W_2(z)$

$w_2(n) = 5x(n) - 0.5w_2(n-1)$ $\qquad = \frac{5z^{-1}X(z)}{1+0.5z^{-1}}$

$W_2(z) = 5X(z) - 0.5z^{-1}W_2(z) \Rightarrow W_2(z) = \frac{5X(z)}{1+0.5z^{-1}}$

$y_3 = w_1(n) + w_3(n)$ ... *

$w_1(n) = 2x(n) + 0.8w_3(n) \Rightarrow W_1(z) = 2X(z) + 0.8W_3(z)$

$w_3(n) = w_1(n-1) \Rightarrow W_3(z) = z^{-1}W_1(z)$

$\Rightarrow W_1(z) = 2X(z) + 0.8z^{-1}W_1(z) \Rightarrow W_1(z) = \frac{2X(z)}{1-0.8z^{-1}}$

$\Rightarrow Y_3 = W_1(z) + W_3(z)$

$= \frac{2X(z)}{1-0.8z^{-1}} + \frac{z^{-1}2X(z)}{1-0.8z^{-1}} = \frac{2X(z)(1+z^{-1})}{1-0.8z^{-1}}$

$Y(z) = Y_1 + Y_2 + Y_3 = 3X(z) + \frac{5z^{-1}X(z)}{1+0.5z^{-1}} + \frac{2X(z)(1+z^{-1})}{1-0.8z^{-1}}$